Solving a linear equation with a skew-symmetric coefficient matrix

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1 Generalities

Consider a vector $\boldsymbol{v} \in \mathbb{R}^3$ and the skew-symmetric 3×3 matrix $\boldsymbol{S}$ built with its components,

$$\boldsymbol{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \qquad \Rightarrow \qquad \boldsymbol{S}(\boldsymbol{v}) = \begin{pmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{pmatrix},$$

so that the following properties hold for vector cross products in $\mathbb{R}^3$:

$$\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3 \qquad \Rightarrow \boldsymbol{v}_1 \times \boldsymbol{v}_2 = \boldsymbol{S}(\boldsymbol{v}_1)\boldsymbol{v}_2 = -\boldsymbol{S}(\boldsymbol{v}_2)\boldsymbol{v}_1.$$

The following properties hold for the singular matrix $\boldsymbol{S}$ ($\det \boldsymbol{S} = 0$, always):

1. $\boldsymbol{S}^T(\boldsymbol{v}) = -\boldsymbol{S}(\boldsymbol{v})$, by definition of skew-symmetry;
2. $\boldsymbol{S}(-\boldsymbol{v}) = -\boldsymbol{S}(\boldsymbol{v}) = \boldsymbol{S}^T(\boldsymbol{v})$;
3. $\boldsymbol{S}(\boldsymbol{v})\boldsymbol{v}$ $(= \boldsymbol{v} \times \boldsymbol{v}) = \boldsymbol{0}$, by definition of cross product;
4. $\boldsymbol{S}^2(\boldsymbol{v}) = \begin{pmatrix} -\left(v_y^2 + v_z^2\right) & v_x v_y & v_x v_z \\ v_y v_x & -\left(v_x^2 + v_z^2\right) & v_y v_z \\ v_z v_x & v_z v_y & -\left(v_x^2 + v_y^2\right) \end{pmatrix} = \boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}\,\|\boldsymbol{v}\|^2$, which is a symmetric matrix;
5. $\operatorname{rank} \boldsymbol{S}(\boldsymbol{v}) = 2$ if and only if $\|\boldsymbol{v}\| \neq 0$;
6. the pseudoinverse is $\boldsymbol{S}^{\#}(\boldsymbol{v}) = -\dfrac{1}{\|\boldsymbol{v}\|^2}\,\boldsymbol{S}(\boldsymbol{v}) = \dfrac{1}{\|\boldsymbol{v}\|^2}\,\boldsymbol{S}^T(\boldsymbol{v})$, satisfying all the four defining relations:

$$\boldsymbol{S}\,\boldsymbol{S}^{\#}\boldsymbol{S} = \boldsymbol{S}, \quad \boldsymbol{S}^{\#}\boldsymbol{S}\,\boldsymbol{S}^{\#} = \boldsymbol{S}^{\#}, \quad \left(\boldsymbol{S}\,\boldsymbol{S}^{\#}\right)^T = \boldsymbol{S}\,\boldsymbol{S}^{\#}, \quad \left(\boldsymbol{S}^{\#}\boldsymbol{S}\right)^T = \boldsymbol{S}^{\#}\boldsymbol{S};$$

7. the null space is $\mathcal{N}(\boldsymbol{S}(\boldsymbol{v})) = \operatorname{span}\{\boldsymbol{v}\}$;

8. the projection matrix in the null space is

$$\boldsymbol{P}(\boldsymbol{v}) = \boldsymbol{I} - \boldsymbol{S}^{\#}(\boldsymbol{v})\,\boldsymbol{S}(\boldsymbol{v}) = \boldsymbol{I} + \frac{1}{\|\boldsymbol{v}\|^2}\,\boldsymbol{S}^2(\boldsymbol{v})$$
$$= \boldsymbol{I} + \frac{1}{\|\boldsymbol{v}\|^2}\left(\boldsymbol{v}\,\boldsymbol{v}^T - \boldsymbol{I}\,\|\boldsymbol{v}\|^2\right) = \frac{\boldsymbol{v}\,\boldsymbol{v}^T}{\|\boldsymbol{v}\|^2}\,;$$

9. $\boldsymbol{P}(\boldsymbol{v})\boldsymbol{v} = \boldsymbol{v}$;
10. powers of $\boldsymbol{S}$ are expressed recursively in terms of $\boldsymbol{S}$ and $\boldsymbol{S}^2$ as

$$\boldsymbol{S}^{2k+1}(\boldsymbol{v}) = (-1)^k\|\boldsymbol{v}\|^{2k}\boldsymbol{S}(\boldsymbol{v}), \qquad \boldsymbol{S}^{2(k+1)}(\boldsymbol{v}) = (-1)^k\|\boldsymbol{v}\|^{2k}\boldsymbol{S}^2(\boldsymbol{v}),$$

for $k = 1, 2, \ldots$.

2 Use for contact localization

The core equations (9) to (12) of present concern in our ICRA 2023 paper are revisited here, together with some further development. The purpose is to show that the final result (11)–(12) follows both from an analytic as well as a geometric argument.

Assume that we have detected a collision between a serial manipulator with n links at a configuration $\boldsymbol{q}$ and the external environment (including a human). Moreover, using the properties of the momentum-based residual vector $\boldsymbol{r} \in \mathbb{R}^n$, we have also isolated the link i that is in collision, with $i \in \{1, \ldots, n\}$. Therefore, one has

$$\boldsymbol{r}^T = \begin{pmatrix} * & \ldots & * & r_i & 0 & \ldots & 0 \end{pmatrix}, \qquad r_i \neq 0, \tag{1}$$

with $r_j = 0$ for $j = i+1, \ldots, n$ and 'don't care' values $(*)$ for the components of $\boldsymbol{r}$ having index $j < i$.

The mapping from an external force $\boldsymbol{f}_i \in \mathbb{R}^3$, acting at the origin of the known kinematic frame RF_i attached to link i, and an external momentum $\boldsymbol{m}_i \in \mathbb{R}^3$, applied to the link i as a whole, to the resulting joint torque $\boldsymbol{\tau} \in \mathbb{R}^n$ is given by

$$\boldsymbol{\tau} = \boldsymbol{J}_i^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix} = \boldsymbol{J}_{Li}^T(\boldsymbol{q})\,\boldsymbol{f}_i + \boldsymbol{J}_{Ai}^T(\boldsymbol{q})\,\boldsymbol{m}_i, \tag{2}$$

where

$$\boldsymbol{J}_i(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_{Li}(\boldsymbol{q}) \\ \boldsymbol{J}_{Ai}(\boldsymbol{q}) \end{pmatrix}$$

is the $6 \times n$ geometric Jacobian matrix for link i, which is known from the (partial) kinematics of the robot arm. For a serial manipulator, the last $n-i$ columns of $\boldsymbol{J}_i$ are zero. Thus, the last $n-i$ components of $\boldsymbol{\tau}$ will be zero as well. In dynamic conditions, the residual vector $\boldsymbol{r}$ is a first-order filtered version (or, a 'proxy') of $\boldsymbol{\tau}$, and this motivates the structure in (1). Therefore, under a full rank condition for the geometric Jacobian $\boldsymbol{J}_i$, one can replace $\boldsymbol{\tau}$ with $\boldsymbol{r}$ in (2) and solve for

$$\begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix} = \boldsymbol{J}_i^{T\,\#}(\boldsymbol{q})\,\boldsymbol{r}, \tag{3}$$

which is eq. (9) in the ICRA 2023 paper. Out of singularities (or rank deficiencies), if the index of the colliding link is $i = 6$, the value obtained from (3) is unique (being $\boldsymbol{J}_i^{T\,\#} = \boldsymbol{J}_i^{-T}$); when $i > 6$, we have $\boldsymbol{J}_i^{T\,\#} = \left(\boldsymbol{J}_i \boldsymbol{J}_i^T\right)^{-1} \boldsymbol{J}_i$, and the solution minimizes the norm of the (possibly zero) error

$$\boldsymbol{e} = \boldsymbol{\tau} - \boldsymbol{J}_i^T(\boldsymbol{q}) \begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix}$$

for the overdetermined system (2) (and among the solutions yielding the minimum error in norm, the one having minimum norm); finally, when $i < 6$ (and the rank of $\boldsymbol{J}_i$ is thus equal to i), there are infinite solutions to the underdetermined system (2) and the one in (3) has a minimum norm. Note that in this case, denoting by $\bar{\boldsymbol{J}}_i$ is the $6 \times i$ submatrix with the first $i < 6$ (full rank) columns of $\boldsymbol{J}_i$, the pseudoinverse solution (3) can be written more explicitly as

$$\begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix} = \bar{\boldsymbol{J}}_i(\boldsymbol{q}) \left(\bar{\boldsymbol{J}}_i^T(\boldsymbol{q})\, \bar{\boldsymbol{J}}_i(\boldsymbol{q})\right)^{-1} \bar{\boldsymbol{r}}^{[i]},$$

where $\bar{\boldsymbol{r}}^{[i]} \in \mathbb{R}^i$ the vector made by the first i components of $\boldsymbol{r}$. Indeed, we have $\bar{\boldsymbol{r}}^{[i]} \neq \boldsymbol{0}$ —see eq. (1).

For link i, consider the standard transformation of the applied forces and moments in two different reference frames, RF_i and RF_{ci}, attached at different points of the rigid body (and with the same relative orientation). The second frame is placed at the point P_{ci} on the link i where the collision with an (unknown) external force $\boldsymbol{f}_{ext} \in \mathbb{R}^3$ occur, while an (unknown) external moment $\boldsymbol{m}_{ext} \in \mathbb{R}^3$ is applied to the entire body. The point P_{ci} is localized by the (unknown) position vector $\boldsymbol{p}_{ci} \in \mathbb{R}^3$ from the origin of frame RF_i. Thus, we have

$$\begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{S}(\boldsymbol{p}_{ci}) & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{f}_{ext} \\ \boldsymbol{m}_{ext} \end{pmatrix}, \tag{4}$$

where the left-hand side is computed by (3) and the right-hand side contains all unknown quantities. In such a transformation, the force remains the same while the mapping between the moments will include also the additional term

$$\boldsymbol{p}_{ci} \times \boldsymbol{f}_{ext} = \boldsymbol{S}(\boldsymbol{p}_{ci}) \boldsymbol{f}_{ext}.$$

Assume now that only an external force $\boldsymbol{f}_{ext} \in \mathbb{R}^3$ is involved in the collision at point P_{ci}, while the external momentum is $\boldsymbol{m}_{ext} = \boldsymbol{0}$. Then, eq. (4) simplifies to

$$\begin{pmatrix} \boldsymbol{f}_i \\ \boldsymbol{m}_i \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{S}(\boldsymbol{p}_{ci}) \end{pmatrix} \boldsymbol{f}_{ext}, \tag{5}$$

which is eq. (10) in the paper. Indeed, the top part implies that $\boldsymbol{f}_{ext} = \boldsymbol{f}_i$, which has been already computed via (3). Knowing thus $\boldsymbol{f}_{ext}$, the bottom part can be rewritten as

$$\boldsymbol{S}(\boldsymbol{p}_{ci}) \boldsymbol{f}_{ext} = -\boldsymbol{S}(\boldsymbol{f}_{ext})\, \boldsymbol{p}_{ci} = \boldsymbol{S}^T(\boldsymbol{f}_{ext})\, \boldsymbol{p}_{ci} = \boldsymbol{m}_i. \tag{6}$$

This is a linear system of three equations in the three unknown components of vector $\boldsymbol{p}_{ci}$, with a coefficient matrix that is skew-symmetric (having rank 2).

Thus, there are an infinite number of solutions, that can all be written in terms of a particular solution (e.g., the one with minimum norm) and a term in the null space of $\boldsymbol{S}^T$ (which is also the null space of $\boldsymbol{S}$).

Therefore, we solve eq. (6) as

$$\boldsymbol{p}_{ci} = -\boldsymbol{S}^{\#}(\boldsymbol{f}_{ext})\,\boldsymbol{m}_i + \Big(\boldsymbol{I} - \Big(-\boldsymbol{S}^{\#}(\boldsymbol{f}_{ext})\Big)\,(-\boldsymbol{S}(\boldsymbol{f}_{ext})\Big)\,\boldsymbol{p}_0, \tag{7}$$

where $\boldsymbol{p}_0 \in \mathbb{R}^3$ is a generic position vector from the origin of frame RF_i. Substituting in (7) the expression of the pseudoinverse of $\boldsymbol{S}$ (and simplifying signs), we obtain

$$\boldsymbol{p}_{ci} = \boldsymbol{S}(\boldsymbol{f}_{ext})\,\frac{\boldsymbol{m}_i}{\|\boldsymbol{f}_{ext}\|^2} + \frac{\boldsymbol{f}_{ext}\,\boldsymbol{f}_{ext}^T}{\|\boldsymbol{f}_{ext}\|^2}\,\boldsymbol{p}_0. \tag{8}$$

The first term in (8) is the position vector of minimum norm that generates, together with $\boldsymbol{f}_{ext}$, the moment $\boldsymbol{m}_i$ at the origin of the reference frame RF_i:

$$\boldsymbol{p}_{c,d} = \boldsymbol{S}(\boldsymbol{f}_{ext})\,\frac{\boldsymbol{m}_i}{\|\boldsymbol{f}_{ext}\|^2} = \frac{1}{\|\boldsymbol{f}_{ext}\|}\cdot\frac{\boldsymbol{f}_{ext}\times\boldsymbol{m}_i}{\|\boldsymbol{f}_{ext}\|},$$

which is the actual expression of the solution value in eq. (11) of the paper. Vector $\boldsymbol{p}_{c,d}$ is orthogonal to both $\boldsymbol{f}_{ext}$ and $\boldsymbol{m}_i$ and its norm is the minimum distance between the *line of action* of $\boldsymbol{f}_{ext}$ and the origin of RF_i. The point P_{cd} with position $\boldsymbol{p}_{c,d}$ is typically *not* on the surface of the link, so it cannot coincide with P_{ci}. Nonetheless, it belongs to the line of action of the force $\boldsymbol{f}_{ext}$.

On the other hand, the second term is a position vector that is always aligned with the line of action of $\boldsymbol{f}_{ext}$ (and normal to $\boldsymbol{p}_{c,d}$. It can be rewritten as

$$\boldsymbol{p}_{c,n}(\lambda) = \lambda\,\frac{\boldsymbol{f}_{ext}}{\|\boldsymbol{f}_{ext}\|}, \qquad \text{with } \lambda = \frac{1}{\|\boldsymbol{f}_{ext}\|}\cdot\boldsymbol{f}_{ext}^T\,\boldsymbol{p}_0 \in \mathbb{R},$$

which is the second term in eq. (12) of the paper. By varying $\lambda \in \mathbb{R}$, namely the projection of the vector $\boldsymbol{p}_0$ along the line of action of $\boldsymbol{f}_{ext}$, we find in general two points of intercept with the surface of the link, one for some value $\lambda = \lambda_{push}$, corresponding to $\boldsymbol{f}_{ext}$ pushing against the link, and the other for a different value $\lambda = \lambda_{pull}$, corresponding to $\boldsymbol{f}_{ext}$ pulling the link. Depending on the context, one can select one or the other solution. For instance, for the more common case of pushing, the estimated position of the collision point P_{ci} is

$$\boldsymbol{p}(\lambda_{push}) = \boldsymbol{p}_{c,d} + \lambda_{push}\,\frac{\boldsymbol{f}_{ext}}{\|\boldsymbol{f}_{ext}\|},$$

which is eq. (12) in the paper.